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ON THE DYNAMICS OF A SOLID ON AN ABSOLUTELY ROUGH PLANE*

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An attempt is made to find a theoretical basis for some dynamic effects discovered experimentally in one problem of solid body dynamics on a plane, namely, the problem of the motion of the "celtic stone" /1—4/. The main attention is given to oscillations of a solid close to the equilibrium position or steady rotation. The motion is assumed to occur without friction and the supporting plane is fixed. Small oscillations of the body are briefly considered in the neighbourhood of its steady rotation about the vertical. An approximate system of equations is obtained which describes non-linear oscillations of the body in the vicinity of its equilibrium position on a plane and a complete analysis is given. The results of the investigation agree with experimental observations /1,3/ of the changes in the direction of rotation the celtic stone about the vertical without any external action, and the origin of rotation in any direction due to oscillations about the horizontal axis.

The dynamics of the celtic stone were first investigated in /2/. It was shown that the rotational stability of the body about the vertical depends on the direction of rotation, and, it was also concluded that it is possible for the body to rotate about the vertical due to its oscillation about the horizontal axis. A rigorous solution of the problem of the stability of the body about the vertical in the absence of slip is given in /5—7/. Numerous experimental conclusions on the motion of the celtic stone are confirmed by numerical integration of the equations of motion in /8/. In /4/ an abstract mathematical model of celtic stone was proposed without analyzing its correspondence to a real solid on a plane.**

1. Let a heavy solid under the action of an initial shock perform a motion in which it rests on one point of its convex surface on an absolutely rough stationary horizontal plane. Let $Oxyz$ be a stationary system of coordinates with origin at the point O of the supporting plane $z=0$. The Oz axis is directed vertically upward. The system of coordinates $G\xi\eta\zeta$, whose axes are directed along the principal central axes of inertia of the body is attached to the body. We take the three Euler angles and the two coordinates x and y of the body's centre of mass in the system $Oxyz$. The third coordinate z of the centre of mass is the distance of that centre from the support plane taken with a positive or negative sign depending on whether the centre of mass lies above (as in Fig.1, which is typical for the celtic stone) or below the support plane. The heavy solid on a fixed horizontal plane is a non-holonomic Chaplygin's system /10/. The differential equations of motion in Chaplygin's form define the motion of the solid relative to its centre of mass, and can be considered independently of the equations of non-integrable couplings that express the absence of slip.

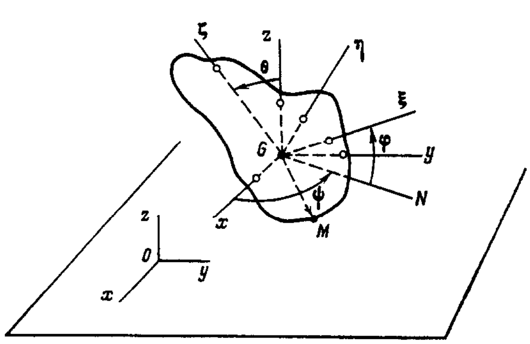

Fig.1

It can be shown that the angle of precession ψ does not appear in the equations of motion /7/, and that they have the particular solution

$$\theta=\pi/2, \quad \varphi=0, \quad \psi^{\cdot}=\omega=\mathrm{const} \qquad (1.1)$$

which corresponds to rotation of the body at an arbitrary constant angular velocity ω about the axis $G\eta$ in a vertical position. Let x_1, x_2, x_3 be the perturbations of the quantities $\theta, \varphi, \psi^{\cdot}$. The equations of perturbed motion are

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**When this paper was in press, the author became aware of the paper by Pascal /9/ which also deals with this problem. The main results obtained in /9/ follow from Sect.4 of the present paper.

$$
\begin{aligned}
&(A + mh^2)\, x_1^{\cdot\cdot} = mlh\omega x_1^{\cdot} - [(A + C - B) + 2mh^2 - \\
&\quad ml_1 h]\, \omega x_2^{\cdot} + [(C - B)\, \omega^2 + m\,(h - l_2)\,(g + \omega^2 h)]\, x_1 - \\
&\quad ml\,(g + \omega^2 h)\, x_2 + X_1 \\
&(C + mh^2)\, x_2^{\cdot\cdot} = [(A + C - B) + 2mh^2 - ml_2 h]\, \omega x_1^{\cdot} - \\
&\quad mlh\omega x_2^{\cdot} - ml\,(g + \omega^2 h)\, x_1 + [(A - B)\, \omega^2 + \\
&\quad m\,(h - l_1)\,(g + \omega^2 h)]\, x_2 + X_2 \\
&Bx_3^{\cdot} = X_3
\end{aligned} \tag{1.2}
$$

$$
\begin{aligned}
&X_1 = \{mlhx_1^{\cdot} - [(A + C - B) + 2mh^2 - ml_1 h]\, x_2^{\cdot} + \\
&\quad 2\omega\, [(C - B) + mh\,(h - l_2)]\, x_1 - 2mlh\omega x_2\}\, x_3 + F_1 \\
&X_2 = \{[(A + C - B) + 2mh^2 - ml_2 h]\, x_1^{\cdot} - mlhx_2^{\cdot} - \\
&\quad 2mlh\omega x_1 + 2\omega\, [(A - B) + mh\,(h - l_1)]\, x_2\}\, x_3 + F_2 \\
&X_3 = \{mlhx_1 - [(A) - B) + mh\,(h - l_1)]\, x_2\}\, x_1^{\cdot\cdot} + \{[C + \\
&\quad mh(h - l_2)]\, x_1 - mlhx_2\}x_2^{\cdot\cdot} + (B + C - A)\, x_1^{\cdot} x_2^{\cdot} + m\,(3h - \\
&\quad r_1 - r_2)\, l\omega\, (x_1^{\cdot} x_2 + \\
&\quad x_1 x_2^{\cdot}) - \omega\, [2\,(C - B) + m\,(2h^2 - 3l_2 h + r_1^2 \cos^2\alpha + \\
&\quad r_2^2 \sin^2\alpha]\, x_1 x_1^{\cdot} - \\
&\quad \omega\, [2\,(A - B) + m\,(2h^2 - 3l_1 h + r_1^2 \sin^2\alpha + r_2^2 \cos^2\alpha)]x_2 x_2^{\cdot} \\
&l = (r_2 - r_1) \sin\alpha \cos\alpha, \quad l_1 = r_1 \sin^2\alpha + r_2 \cos^2\alpha \\
&l_2 = r_1 \cos^2\alpha + r_2 \sin^2\alpha
\end{aligned} \tag{1.3}
$$

where m is the mass of the body, g the acceleration due to gravity, A, B, C are the moments of inertia of the body about the axes $G\xi, G\eta, G\zeta$, h is the distance of the body centre of mass from the support plane, taken with the appropriate sign in the unperturbed motion (1.1), r_1 and r_2 are the principal radii of curvature of the body surface at its point of contract with the plane, and α is the angle between the $G\zeta$ axis and the curvature line corresponding to r_1, read counterclockwise from the $G\zeta$ axis looking along the $G\eta$ axis, which in (1.1) is vertical, toward the origin G. The quadratic forms in $x_i, x_i^{\cdot}$ $(i = 1, 2)$ are denoted by $F_1$and F_2 their explicit form is not required.

2. The characteristic equation of the linearized system of equations of the perturbed motion (1.2) is written in the form /7/

$$
\begin{aligned}
&\lambda\,(P\lambda^4 + Q\omega\lambda^3 + R\lambda^2 + Q\omega^3\lambda + S\,) = 0 \\
&P = (A + mh^2)\,(C + mh^2), \quad Q = mlh\,(A - C) \\
&R = [(A + C - B + 2mh^2)^2 - (A + C - B + 2mh^2)\, mh\,(r_1 + \\
&\quad r_2) + m^2h^2 r_1 r_2]\, \omega^2 - (A + mh^2)\, [(A - B)\, \omega^2 + m\,(h - \\
&\quad l_1)\,(g + \omega^2 h)] - (C + mh^2)\, [(C - B)\, \omega^2 + m\,(h - l_2)\cdot(g + \omega^2 h)] \\
&S = (A - B)\,(C - B)\, \omega^4 + m\,(g + \omega^2 h)\, \omega^2\, [A\,(h - l_2) + \\
&\quad C\,(h - l_1) - B\,(2h - r_1 - r_2)] + m^2\,(g + \omega^2 h)^2\cdot(h - r_1)\,(h - r_2)
\end{aligned} \tag{2.1}
$$

The conditions

$$
(R - P\omega^2)\, \omega^2 - S > 0, \quad S > 0 \tag{2.2}
$$

$$
\omega h\,(A - C)\,(r_2 - r_1) \sin\alpha \cos\alpha > 0 \tag{2.3}
$$

were obtained in /5–7/. When they are satisfied, the motion (1.1) is asymptotically stable relative to perturbations of the quantities $\theta, \theta^{\cdot}, \varphi, \varphi^{\cdot}$. Inequality (2.2) imposes constraints on the mass distribution, the body surface geometry and the magnitude of the angular velocity, while inequality (2.3) imposes constraints on the sign of the angular velocity (the direction of rotation) of the body. If $h > 0$, i.e., in the unperturbed motion the centre of mass of the body lies above the support plane, then in steady rotation the smaller horizontal axis of the central ellipsoid of inertia moves ahead of the line of minimum curvature of the body surface at its point of contact with the plane; when $h < 0$ the pattern is reversed.

It was also shown in /5–7/ that instability occurs if only one of inequalities (2.2) or (2.3) is violated. This implies that when $Q \neq 0$, the steady rotation (1.1) is unstable for fairly small ω, irrespective of its sign, since for small ω inequalities (2.2) are incompatible.

If at least one of the quantities $\omega, h, A - C, r_2 - r_1, \sin 2\alpha$ is zero, then $Q = 0$ and inequality (2.3) is not satisfied. The characteristic equation (2.1) has, as before, a single zero root, and the remaining four satisfy the biquadratic equation. Let the biquadratic equation have two pairs of purely imaginary roots $\pm i\omega_1, \pm i\omega_2 (\omega_1 > \omega_2 > 0)$. Then, the motion is stable in the linear approximation. Let us consider the roots of Eq. (2.1) for small $Q\omega$. The calculations show that to a first approximation in $Q\omega$, the roots $\pm i\omega_j$ $(j = 1, 2)$ in addition to corrections to their imaginary parts, also have real parts $\varkappa_j$ $(j = 1, 2)$

$$\varkappa_1 = \frac{Q\omega(\omega^2 - \omega_1^2)}{2(A + mh^2)(C + mh^2)(\omega_1^2 - \omega_2^2)} \tag{2.4}$$
$$\varkappa_2 = \frac{Q\omega(\omega_2^2 - \omega^2)}{2(A + mh^2)(C + mh^2)(\omega_1^2 - \omega_2^2)}$$

Suppose $Q\omega > 0$, i.e. inequality (2.3) is satisfied. It then follows from (2.4) that for $\omega_2^2 < \omega^2 < \omega_1^2$ we have $\varkappa_j < 0$ $(j = 1, 2)$ and the small oscillations of the body close to its stationary rotation (1.1), are exponentially damped. If $0 < \omega^2 < \omega_2^2$, then $\varkappa_1 < 0, \varkappa_2 > 0$ and the high-frequency oscillations (or frequency ω_1) are exponentially damped, and the low-frequency oscillations (of frequency ω_2) increase exponentially. If, however, $\omega^2 > \omega_1^2$, then conversely, the low-frequency oscillations are damped, and the high-frequency oscillations increase. When $Q\omega < 0$ the development of small oscillations is the opposite.

3. Let $\omega = 0$ in (1.1), i.e. the body rests on the plane on a single point of the $G\eta$ axis which is vertical. The necessary and sufficient condition for this equilibrium position to be stable is, according to /5/, inequalities $r_1 > h$, $r_2 > h$. Assuming that this condition is satisfied, we will consider the motion of the body close to the position of equilibrium.

The equations of perturbed motion have the form

$$(A + mh^2)x_1^{\cdot\cdot} = mg(h - l_2)x_1 - mglx_2 + X_1 \tag{3.1}$$
$$(C + mh^2)x_2^{\cdot\cdot} = -mglx_1 + mg(h - l_1)x_2 + X_2, \qquad Bx_3^{\cdot} = X_3$$

where X_j $(j = 1, 2, 3)$ are the respective functions of (1.3) calculated for $\omega = 0$. We change the variables $x_1, x_2, x_3 \to y_1, y_2, y_3$, in system (3.1) which reduces the first two of its linearized equations to the form corresponding to normal oscillations.

The frequencies Ω_1, Ω_2 $(\Omega_1 > \Omega_2 > 0)$ of normal oscillations satisfy the equation

$$(A + mh^2)(C + mh^2)\Omega^4 - mg[(A + mh^2)(l_1 - h) + (C + mh^2)(l_2 - h)]\Omega^2 + (mg)^2(r_1 - h)(r_2 - h) = 0 \tag{3.2}$$

The reduction to standard coordinates y_1, y_2 is obtained by changing the variables

$$x_1 = u_{11}y_1 + u_{12}y_2, \quad x_2 = u_{21}y_1 + u_{22}y_2, \quad x_3 = y_3 \tag{3.3}$$
$$u_{1j} = k_j mgl, \quad u_{2j} = k_j[(A + mh^2)\Omega_j^2 + mg(h - l_2)]$$
$$k_j = \{(A + mh^2)(mgl)^2 + (C + mh^2)[(A + mh^2)\Omega_j^2 + mg(h - l_2)]^2\}^{-1/2}$$
$$(j = 1, 2)$$

In the linear approximation with respect to x_1, x_2 the equation of the trace of the contact point M on the body surface is

$$\xi(t) = -lx_1(t) - l_1x_2(t), \quad \zeta(t) = l_2x_1(t) + l\,x_2(t)$$

Hence it follows from (3.3) that for the j-th normal oscillation (with frequency Ω_j) the tangent to the trace of point M makes an angle β_j with the $G\zeta_{,,}$ axis calculated from the formula

$$\operatorname{tg}\beta_j = -(lu_{1j} + l_1u_{2j})/(l_2u_{1j} + lu_{2j}) \quad (j = 1, 2)$$

Hence we see what the perturbations x_1, x_2 should be, if the body is to perform high-frequency (of frequency Ω_1) or low-frequency (of frequency Ω_2) small oscillations.

In variables y_1, y_2, y_3 Eqs. (3.3) take the form

$$y_1^{\cdot\cdot} + \Omega_1^2y_1 = (a_1y_1^{\cdot} + a_2y_2^{\cdot})y_3 + G_1 \tag{3.4}$$
$$y_2^{\cdot\cdot} + \Omega_2^2y_2 = (b_1y_1^{\cdot} + b_2y_2^{\cdot})y_3 + G_2$$
$$By_3^{\cdot} = Y_3$$

$$a_j = (c_1u_{22} + c_2u_{12})u_{1j} + (d_1u_{22} + d_2u_{12})u_{2j} \tag{3.5}$$
$$b_j = -(c_1u_{21} + c_2u_{11})u_{1j} - (d_1u_{21} + d_2u_{11})u_{2j}$$
$$(j = 1, 2)$$
$$c_1 = -\frac{mlh}{(A + mh^2)\Delta}, \quad c_2 = \frac{(A + C - B) + 2mh^2 - ml_2h}{(C + mh^2)\Delta}$$
$$d_1 = \frac{(A + C - B) + 2mh^2 - ml_1h}{(A + mh^2)\Delta}, \quad d_2 = -\frac{mlh}{(C + mh^2)\Delta}$$
$$\Delta = k_1k_2mgl(A + mh^2)(\Omega_1^2 - \Omega_2^2)$$

$$Y_3 = -(u_{11}\Omega_1^2y_1 + u_{12}\Omega_2^2y_2)\{mlh(u_{11}y_1 + u_{12}y_2) - [(A - B) + mh(h - l_1)](u_{21}y_1 + u_{22}y_2)\} - (u_{21}\Omega_1^2y_1 + u_{22}\Omega_2^2y_2)\{[C + mh(h - l_2)](u_{11}y_1 + u_{12}y_2) - mlh(u_{21}y_1 + u_{22}y_2)\} + (B + C - A)(u_{11}y_1^{\cdot} + u_{12}y_2^{\cdot})(u_{21}y_1^{\cdot} + u_{22}y_2^{\cdot}) \tag{3.6}$$

As in (3.1), terms of higher order than the second relative to perturbations have been omitted in system (3.4). We denote by G_1, G_2 the quadratic forms of the variables y_j; $y_j^{\cdot}$ ($j = 1, 2$).

To investigate the non-linear system (3.4) we reduce it to the normal form /11/, and, first, make the change of variables

$$y_1 = \frac{z_1 - z_3}{2i\Omega_1}, \quad y_2 = \frac{z_2 - z_4}{2i\Omega_2}, \quad y_1^{\cdot} = \frac{z_1 + z_3}{2}, \quad y_2^{\cdot} = \frac{z_2 + z_4}{2}, \tag{3.7}$$
$$y_3 = z_5$$

In variables z_k ($k = 1, \ldots, 5$) the linear part of system (3.4) has a diagonal form and the derivation of its normal form reduces to separating resonant terms from the non-linearities on the right-hand sides of the transformed system (3.4). When $\Omega_1 \neq 2\Omega_2$, the structure of the normal form is independent in Eqs. (3.4) of the quadratic forms G_j ($j = 1, 2$) or, what is the same, of the quadratic forms F_j in (1.3). Assuming that $\Omega_1 \neq 2\Omega_2$, we obtain the following normal form of system (3.4) written in complex variables:

$$z_1^{\cdot} = i\Omega_1 z_1 + c_{10001} z_1 z_5, \quad z_2^{\cdot} = i\Omega_2 z_2 + c_{01001} z_2 z_5 \tag{3.8}$$
$$z_3^{\cdot} = -i\Omega_1 z_3 + c_{00101} z_3 z_5, \quad z_4^{\cdot} = -i\Omega_2 z_4 + c_{00011} z_4 z_5$$
$$z_5^{\cdot} = c_{10100} z_1 z_3 + c_{01010} z_2 z_4$$
$$c_{10001} = c_{00101} = a_1/2, \quad c_{01001} = c_{00011} = b_2/2 \tag{3.9}$$
$$c_{10100} = mh\,(r_2 - r_1)\,\Omega_1^2\,[(u_{21}^2 - u_{11}^2)\sin\alpha\cos\alpha - u_{11}u_{21}\cos 2\alpha]/(2B)$$
$$c_{01010} = mh\,(r_2 - r_1)\,\Omega_2^2\,[(u_{22}^2 - u_{12}^2)\sin\alpha\cos\alpha - u_{12}u_{22}\cos 2\alpha]/(2B)$$

Introducing real polar coordinates in conformity with the formulas

$$z_1 = \rho_1(\cos\sigma_1 + i\sin\sigma_1), z_2 = \rho_2(\cos\sigma_2 + i\sin\sigma_2),$$
$$z_3 = \bar{z}_1,\ z_4 = \bar{z}_2,\ z_5 = \rho_3$$

and carrying out some operations using formulas (3.3) and (3.5), and the frequency equation (3.2), we obtain the normalized system of equations of perturbed motion which is then split into two independent subsystems

$$\rho_1^{\cdot} = -a\Omega_1^2\rho_1\rho_3, \quad \rho_2^{\cdot} = a\Omega_2^2\rho_2\rho_3, \quad B\rho_3^{\cdot} = a\,(\Omega_1^4\rho_1^2 - \Omega_2^4\rho_2^2) \tag{3.10}$$
$$\sigma_1^{\cdot} = \Omega_1, \quad \sigma_2^{\cdot} = \Omega_2 \tag{3.11}$$
$$a = \frac{(A - C)\,mh\,(r_2 - r_1)\sin\alpha\cos\alpha}{2\,(A + mh^2)(C + mh^2)(\Omega_1^2 - \Omega_2^2)}$$

Terms of order higher than the second in (3.10), and those higher than the first in ρ_k ($k = 1, 2, 3$) in (3.11) have been omitted.

4. In the ε-neighbourhood of the equilibrium position, the right-hand sides of Eqs. (3.10) and (3.11) differ from the respective right-hand sides of the exact equations of perturbed motion by quantities of order ε^3 and ε^2, respectively. The solutions of the exact equations are approximated by the solutions of system (3.10) and (3.11) with an error of ε^2 for ρ_k and of order ε for σ_j in a time interval of order ε^{-1}. Restricting the calculations to this accuracy, we shall consider the approximate system (3.10), (3.11) instead of the complete equations of perturbed motion.

Equations (3.11) are readily integrable. We obtain $\sigma_j(t) = \Omega_j t + \sigma_j(0)$ ($j = 1, 2$). System (3.10) has the integrals

$$\Omega_1^2\rho_1^2 + \Omega_2^2\rho_2^2 + B\rho_3^2 = B\mu^2 \quad (\mu > 0) \tag{4.1}$$
$$\rho_1^{\varkappa}\rho_2 = \nu \quad (\varkappa = \Omega_2^2/\Omega_1^2) \tag{4.2}$$

where μ and ν are constants determined by the initial conditions.

The trajectories of system (3.10) are respresented in Fig.2 in space ρ_1, ρ_2, ρ_3. They lie in the region $\rho_1 \geqslant 0, \rho_2 \geqslant 0$, and represent curves representing the intersections of the surface of the ellipsoid (4.1) and the cylindrical surface (4.2). We use the notation $A_j = B^{1/2}\mu/\Omega_j$ ($j = 1, 2$). For a given constant μ the quantity ν must satisfy the inequalities

$$0 \leqslant \nu \leqslant \nu_* = \varkappa^{-1}(B\mu^2\varkappa/[(1 + \varkappa)\,\Omega_1^2])^{(1+\varkappa)/2}$$

If $\nu > \nu_*$ the motion is impossible. The plane $\rho_1\Omega_1^2 = \rho_2\Omega_2^2$ on which the right-hand side of the third equation of system (3.10) vanishes is shaded. The trajectories are symmetrical about the plane $\rho_3 = 0$. The direction of motion along the trajectories is indicated by the arrows. It is assumed that $a > 0$; when $a < 0$ the motion is in the opposite direction.

Let us consider the properties of the solutions of system (3.10) and their relations with the properties of motion of the solid in the plane. The points $P_1 = (0, 0, \mu)$, $P_2 = (0, 0, -\mu)$, $P_3 = (\rho_1^0, \rho_2^0, 0)$ in Fig.2 denote the equilibrium position of system (3.10). Steady rotations about the vertical are denoted by the points P_1 and P_2, respectively, counterclockwise at an angular velocity μ and clockwise at an angular velocity $-\mu$. Both rotations are unstable, as implied by the linearized equations (3.10), and illustrated in Fig.2.

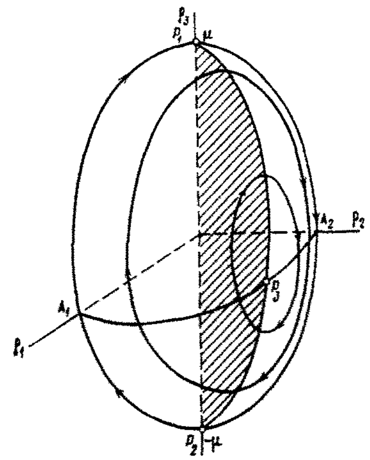

Fig.2

The equilibrium position P_3 corresponds to conditionally periodic, or periodic oscillations of the body, when Ω_1/Ω_2 is a rational number (not equal two, since the case when $\Omega_1 = 2\Omega_2$ is excluded form consideration). Then

$$\rho_1^\circ = \frac{\mu}{\Omega_1}\left(\frac{B\varkappa}{1+\varkappa}\right)^{1/2}, \quad \rho_2^\circ = \frac{\mu}{\Omega_1}\left[\frac{B}{\varkappa(1+\varkappa)}\right]^{1/2}, \quad \nu = \nu_*$$

The effects that are characteristic of celtic stones /1–3/ are not observed: oscillations about the horizontal axes do not induce rotation of the body about the vertical ($\rho_3 \equiv 0$). To investigate the stability of the oscillations we use the Liapunov stability theorem /12/. We construct a function V in the form of the bundle of integrals (4.1) and (4.2). Setting $\rho_1 = \rho_1^\circ + R_1$, $\rho_2 = \rho_2^\circ + R_2$, $\rho_3 = R_3$, we rewrite them in the form

$$V_1 = R_1 + R_2 + \frac{1}{2\rho_1^\circ} R_1^2 + \frac{1}{2\rho_2^\circ} R_2^2 + \frac{B}{2\rho_1^\circ\Omega_1^2} R_3^2 = \text{const}$$

$$V_2 = R_1 + R_2 + \frac{\Omega_2^2 - 1}{2\rho_1^\circ} R_1^2 + \frac{\Omega_1^2}{\rho_2^\circ} R_1R_2 + \frac{\Omega_1^2 - 1}{2\rho_2^\circ} R_2^2 + \cdots = \text{const}$$

The dots in the formula for V_2 denote terms of higher order than the second with respect to perturbations R_1 and R_2. We set $V = V_1 - V_2 + \Omega_1^2 V_2^2/(2\rho_2^\circ)$ and obtain the formula

$$V = \frac{1}{\rho_1^\circ} R_1^2 + \frac{1}{\rho_2^\circ} R_2^2 + \frac{B}{2\rho_1^\circ\Omega_1^2} R_3^2 + \cdots \tag{4.3}$$

Since the function (4.3) is positive definite, the oscillations considered are stable with respect to perturbations ρ_1, ρ_2, ρ_3. This conclusion is illustrated in Fig.2, where the point P_3 is surrounded by closed trajectories lying on the ellipsoid (4.1), as close to that point as desired.

The system of Eqs.(3.10) has the following particular solutions:

$$\rho_1 = 0, \ \rho_2(t) = A_2 \operatorname{sch}[\delta_1(t + e_1)], \ \rho_3 = \mu \operatorname{th}[\delta_1(t + e_1)] \tag{4.4}$$
$$(\delta_1 = -a\mu\Omega_2^2, \ e_1 = \delta_1^{-1} \operatorname{Arth}[\rho_3(0)/\mu])$$

$$\rho_1(t) = A_1 \operatorname{sch}[\delta_2(t + e_2)], \rho_2 = 0, \rho_3 = \mu \operatorname{th}[\delta_2(t + e_2)] \tag{4.5}$$
$$(\delta_2 = a\mu\Omega_1^2, \ e_2 = \delta_2^{-1} \operatorname{Arth}[\rho_3(0)/\mu])$$

in which ρ_1 or ρ_2 are identically equal to zero.

These solutions are represented in Fig.2 by asymptotic trajectories that connect the unstable equilibrium positions P_1 and P_2.

Solution (4.4) corresponds to motions of the body, when it rotates about the vertical and executes low-frequency oscillations. If $\rho_3(0) \leqslant 0$, i.e. at the initial instant the body either is not rotating about the vertical, or is rotating clockwise, then in the course of time the "amplitude" of the oscillations ρ_2 decreases monotonically (when $a > 0$, as in Fig. 2) from the initial value $\rho_2(0)$ to zero, while the angular velocity increases in absolute value. In the limit the body performs pure rotation about the vertical in a clockwise direction at an angular velocity $-\mu$. If, however, $\rho_3(0) > 0$, i.e. at the initial instant the body is rotating counterclockwise, the limit of the body motions is the same as when $\rho_3(0) \leqslant 0$, but the evolution of the motion is entirely different. When $0 < t < t_* = -e_1$, the oscillation amplitude ρ_2 increases monotonically and the body rotates about the vertical counterclockwise at decreasing angular velocity. At the instant $t = t_*$ the angular velocity vanishes and the oscillation amplitude ρ_2 reaches its maximum value A_2. When $t > t_*$, the body already rotates clockwise at an increasing absolute value of angular velocity, and the oscillation amplitude decreases monotonically. Thus when $\rho_3(0) > 0$, during the time of evolution of the motion a change in the direction of rotation of the body about the vertical occurs only once.

Solution (4.5) defines a motion in which the body, while rotating about the vertical, performs high-frequency oscillations. The analysis of the evolution of the motion is similar to the preceding case. The limit motion here is a pure counterclockwise rotation about the vertical at an angular velocity μ. If at the initial instant the body is rotating clockwise about the vertical, then at the instant $t = -e_2$ a change in the direction of rotation occurs. At that instant the oscillation amplitude ρ_1 reaches its maximum value A_1.

Let us now consider solutions of system (3.10) that are different from those of (4.4), (4.5) and from the equilibrium position P_i $(i = 1, 2, 3)$. From the integrals (4.1) and (4.2) we have

$$\rho_2 = \nu\rho_1^{-\varkappa}, \quad \rho_3 = \pm f(\rho_1)$$
$$f(\rho_1) = \frac{[-\Omega_2^2\nu^2 + B\mu^2\rho_1^{2\varkappa} - \Omega_1^2\rho_1^{2(\varkappa+1)}]^{1/2}}{B^{1/2}\rho_1^{\varkappa}} \tag{4.6}$$

Substituting ρ_3 from (4.6) into the first equation of system (3.10) and separating the variables, we obtain

$$\frac{d\rho_1}{\rho_1 f(\rho_1)} = \mp a\Omega_1^2 dt \tag{4.7}$$

If the function $\rho_1(t)$ is determined from (4.7), then $\rho_2(t)$ and $\rho_3(t)$ are calculated using Eqs.(4.6).

It is not possible, in general, to obtain an explicit analytic expression for the function $\rho_1(t)$. However the qualitiative nature of the motion may be directly obtained from the system of Eqs.(3.10). For example, at the initial instant of time let the right-hand side of the third equation of system (3.10) and the quantity ρ_3 be positive. The pattern of motion is as follows (Fig.2). When $t > 0$ the body is rotating counterclockwise about the vertical more and more rapidly (ρ_3 is increasing); the amplitude of the high-frequency oscillations of ρ_1 decreases, and that of the low-frequency oscillations ρ_2 increases. This ultimately results in the right-hand side of the third equation of system (3.10) vanishing; in Fig.2 this corresponds to the instant at which the trajectory intersects the plane $\rho_1\Omega_1^2 = \rho_2\Omega_2^2$. At that instant the angular velocity ρ_3 of rotation about the vertical reaches its maximum and begins to decrease, remaining positive (the body continues to rotate counterclockwise about the vertical, and ρ_1, as before, decreases and ρ_2 increases. This countinues until the angular velocity vanishes. At that instant ρ_1 and ρ_2 reach their minimum and maximum values, respectively, and then ρ_1 begins to increase and ρ_2 to decrease, while the body is already rotating in the opposite direction (clockwise) ($\rho_3 < 0$) at an ever-increasing absolute value of the angular velocity. The decrease of ρ_2 and increase of ρ_1 results in the right-hand side of the third equation of system (3.10) again vanishing (in Fig.2 the trajectory again intersects the plane $\rho_1\Omega_1^2 = \rho_2\Omega_2^2$, but in the region of negative ρ_3). At that instant the angular veloscity of the body in clockwise rotation reaches its maximum in absolute value, after which the rotation of the body begins to slow down, ρ_1 continues to increase, and ρ_2 continues to decrease. This pattern continues until ρ_3 vanishes, when ρ_1 and ρ_2 reach their maximum and minimum values respectively, and the body changes its rotation from clockwise to counterclockwise. The pattern of motion is subsequently periodically repeated. The closed trajectory in Fig.2 corresponds to the cycle of motion described. The period of oscillations may be determined using Eq.(4.7).

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